# Episode 403: Orbital motion

In this episode, students will learn how to combine concepts learned in the study of circular motion with Newton's Law of Universal Gravitation to understand the (circular) motion of satellites. While real orbits are very rarely circular, they should still find it satisfying to see the relative simplicity with which the laws they have learned lead to an explanation of Kepler's 3rd law, which had previously been only an empirical law.

We will start with a thought-experiment, often attributed to Newton, in which students will learn that astronauts orbiting the Earth are not actually weightless, but feel as though they are because they are in constant free-fall.

There are some applets available to aid discussion in this episode. If it is at all possible, students should try to play around with these applets themselves too.

## Summary

**Discussion: Newton's cannonball. (10 minutes)**

**Discussion: Kepler's laws (20 minutes)**

**Discussion: Geostationary orbits (5 minutes)**

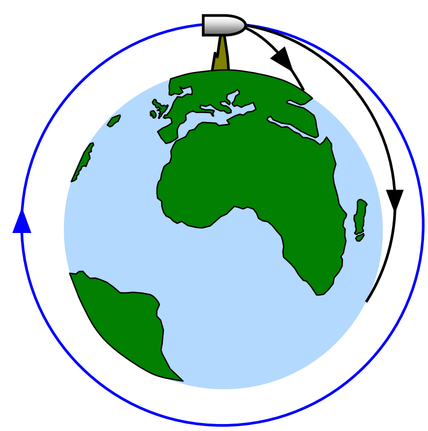
**Worked examples: Orbital motion (25 minutes)**

**Student questions (15 minutes, 25 minutes)**

## Discussion

## Newton's cannonball

What is the motion of a cannonball fired horizontally on the Earth's surface, neglecting air resistance? (If you fire a cannonball horizontally (neglecting air resistance), it will travel some distance before it strikes the ground. If you fire it much faster (again horizontally) then it will travel much further before hitting the ground.)



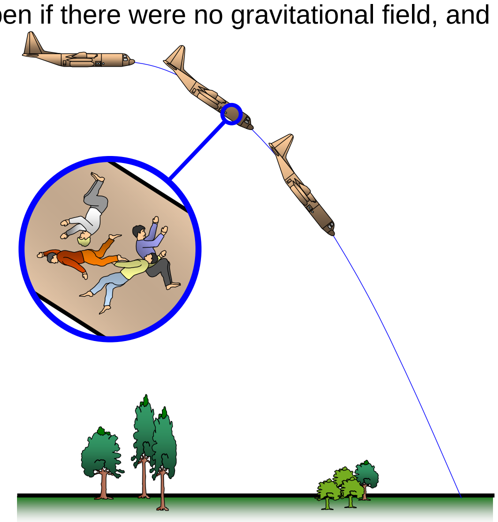
(Newton asked what would happen if you fired faster again. He reasoned that the cannonball would maintain a constant height from the surface of the Earth – in other words, it would move in a circular orbit.)

Newton's original diagram can be found at the website below, together with an applet to illustrate the ideas (instructions on the webpage):

http://galileo.phys.virginia.edu/classes/109N/more_stuff/Applets/newt/newtmtn.html

The cannonball is fired horizontally from the top of a mountain at a user-definable initial speed. For speeds that are too slow, the cannonball falls towards the Earth's surface. Above a certain speed (about 15 500 mph) the cannonball orbits the Earth (the applet stops it after one orbit). At a speed of just over 16 000 mph this orbit is circular – i.e. at a fixed height above the Earth's surface.

How would you feel if you were orbiting like the cannonball? (Well, you would be falling the whole time, whilst moving horizontally at the same time. Imagine falling in a lift with the cable cut. You'd have butterflies in your stomach and if you held a ball out in front of you and let go it wouldn't move relative to you. This is exactly what would happen if there were no gravitational field, and it is in this sense that you would feel weightless. When astronauts in an orbiting spacecraft are said to be weightless, it is not actually true – they are still in (a relatively strong part of) the Earth's gravitational field, and so they have a weight, but they just don't feel it because both they and the spacecraft surrounding them are constantly falling towards the centre of the Earth (whilst also moving tangentially around the orbit).)


(resourcefulphysics.org)

So, launching a satellite into a circular orbit requires it being set off with the right tangential speed for the height at which you want it. The following applet allows you to play around with initial speeds/directions to see what would happen. It's quite surprising to some students to realise that at a particular height, there is a range of speeds and directions that you can start the satellite at, for which it will be trapped in orbit (although elliptical, not circular). Instructions are on the webpage.

http://webphysics.ph.msstate.edu/jc/library/5-2c/index.html

## Discussion:

## Kepler's laws

Kepler discovered three laws of planetary motion based on the painstakingly recorded observations of Brahe in the early 17$^{th}$ century. These laws were however empirical – that is they fitted the patterns observed but had no physical explanation. Newton later managed to explain the laws with his laws of motion and Law of Universal Gravitation. Kepler's three laws were:

All of the planets move in elliptical orbits with the Sun at one focus.

The radius vector (imaginary line between the Sun and the planet) sweeps out equal areas in equal times.

The square of the period of a planet's orbit is directly proportional to the cube of its mean distance from the Sun.

These laws apply for all satellite motions (not just for the planets around the Sun). Remember that circular orbits are just a special case of elliptical orbits.

Each of these laws can be demonstrated with the applet on the webpage below (instructions on the webpage):

http://www.phy.ntnu.edu.tw/java/Kepler/Kepler.html?CFID=3179838&CFTOKEN=86948226

We shall concentrate on circular orbits at constant speed only (elliptical orbits are well beyond the scope of A-level courses). The first two of Kepler's laws are trivial for circular orbits. An ellipse has two foci normally but in the case of a circle they are coincident and at the centre of the circle. Also if the planet moves at constant speed in a circle then the radius vector has to sweep out equal areas in equal times. So we shall concentrate on deriving the third law for circular orbits.

What is the only force acting on a planet orbiting the Sun? (The gravitational pull from the Sun, neglecting the negligible pulls from other objects in the solar system and beyond.)

And if the planet is moving at constant speed in a circle it must have centripetal force acting on it.

What is the expression for centripetal force? $mv^2/r$

For the gravitational force? $Gm_1m_2/r^2$

Equating these expressions, we have

$$GMm/r^2 = mv^2/r$$

(with *M* being the mass of the Sun and *m* being the mass of the planet).

But the speed *v* can be calculated as distance travelled in one orbit ($2\pi r$) divided by the time taken, *T*:

$$v = 2\pi r/T$$

Plugging this into the previous equation, and cancelling the m terms on both sides gives us:

$$GM/r^2 = 4\pi^2 r/T^2$$

Rearranging again gives:

$$T^2 = (4\pi^2/GM)\, r^3$$

Or in other words, since $(4\pi^2/GM)$ is constant, the square of the period of orbit is proportional to the cube of the radius of the orbit – Kepler's third law. A powerful result proved simply by the laws of motion and gravitation. This law shows that the further away a planet is, the longer its period of orbit (but in a non-linear way).

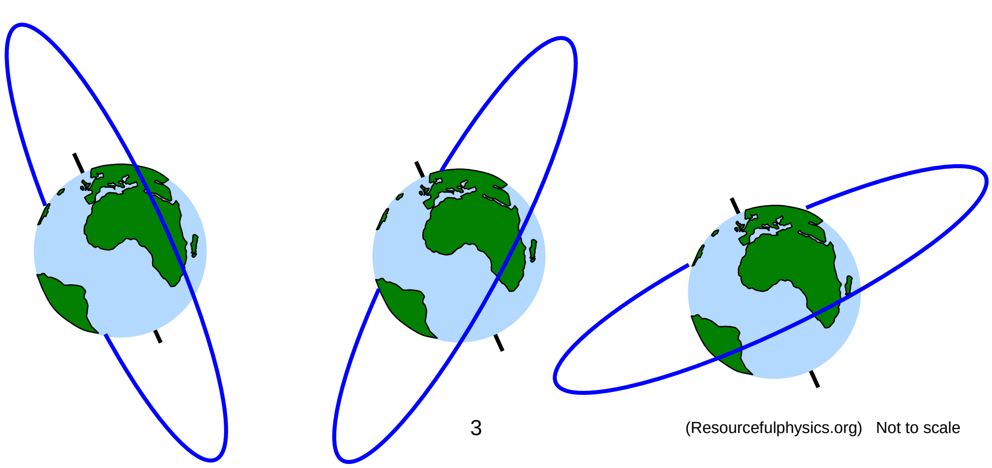
Although this law (as are all of Kepler's laws) was originally discovered for the motion of the planets, it works equally well for all satellite motion – for example the motion of the Moon and artificial satellites around the Earth. We just need to realise that the actual M in the constant $(4\pi^2/GM)$ is the mass of the Earth in that case (or the mass of whichever body is being orbited).

## Discussion:

## Geostationary orbits

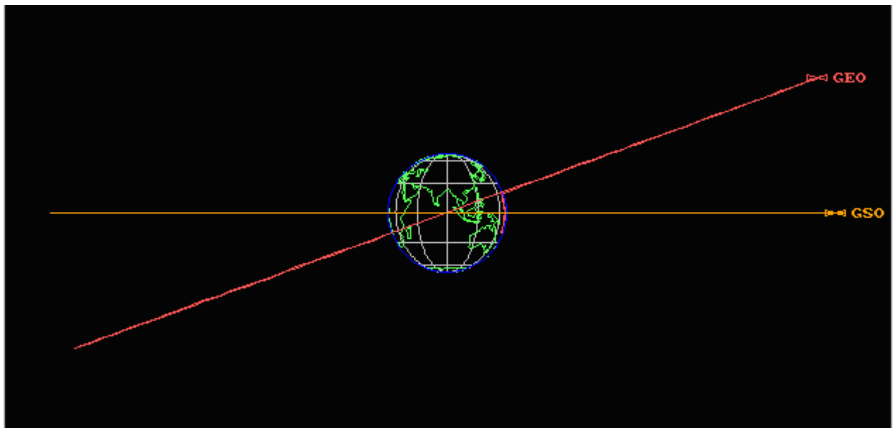
We can apply this idea to geostationary orbits. We know from Kepler's third law that the further away a satellite is from the body it is orbiting, the longer its orbital period.

If an orbiting satellite had a period of 24 hours, and you saw it overhead at, say 10.00 am, when would you next see it overhead? (Because both the Earth would have completed one rotation in the same time it took the satellite to complete one orbit, it would next be overhead at 10.00 am the next day. Such a satellite is said to be *geosynchronous*.)



A difficult question – if you wanted the satellite to remain directly overhead (i.e. above a fixed point on the Earth) at all times (not just once per day) where on the Earth would you have to be? [All satellites (in circular orbits) orbit around the centre of the Earth. The only points on the Earth's surface that orbit around the centre of the Earth are those on the equator. Thus, you would have to be on the equator.]

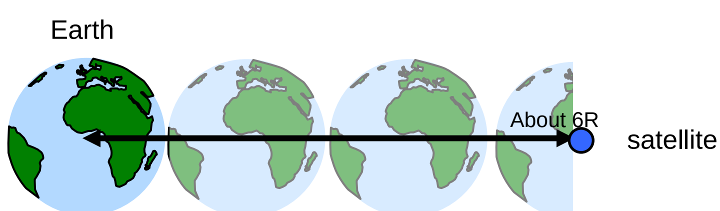
If a satellite has a period of 24 hours and orbits above the equator such that it always appears to be above one point on the equator, it is known as a *geostationary* satellite, and its orbit is a geostationary orbit. A geostationary orbit has a radius of around 42 000 km (over 6 times the radius of the Earth) – i.e. at a height above the Earth's surface of around 36 000 km. (These figures will be calculated in the worked examples that follow).




Picture taken from http://celestrak.com/columns/v04n07/

The orbit labelled GEO above is geosynchronous, but not geostationary, because the satellite would appear from the equator to wander first north, and then south and then back again over a 24 hour period.

The orbit labelled GSO is geostationary.




Geostationary satellites are predominantly used for communications. Satellite TV companies use geostationary satellites to cover a constant area on the Earth's surface – hence you point your satellite dish receiver in the direction of the geostationary satellite. 3 geostationary satellites placed into orbit 120 degrees apart above the equator would be able to cover the entire Earth (except for very near the poles). Because geostationary satellites have to be launched so high (other satellites orbit as low as a few hundred km), the energy and costs required for launching a satellite into geostationary orbit are high.

## Worked examples:

## Orbital motion

These examples look at orbital motion under gravity. They can be tackled independently by most students, but solutions are provided if you wish to use them as worked examples.

TAP 403-1: Worked examples – orbital motion

## Student questions

TAP 403-2: Using Kepler's third law

TAP 403-3: Changing orbits

# TAP 403-1: Worked examples – Orbital Motion

## Student sheet

Data required:

$G$ = 6.67 × $10^{-11}$ N $m^2$ $kg^{-2}$

mass of the Earth = 6.0 × $10^{24}$ kg

radius of the Earth = 6.4 × $10^6$ m

mass of Jupiter = 1.9 × $10^{27}$ kg

radius of Jupiter = 7.2 × $10^7$ m

Jupiter's day length = 10 hours

1) What is the only force acting on a single planet orbiting a star? Write down an expression for this force. If the planet moves in a circular orbit of radius *r*, at constant speed *v*, write down an expression for this speed in terms of the period *T* of the orbit.

Because the orbit is circular, the planet must experience a centripetal force of size $mv^2/r$. Use this fact and the 2 expressions you have written down to prove Kepler's third law, which states that the square of the time period of the planet's orbit is proportional to the cube of the radius of the orbit.

2) Use Kepler's third law, $T^2 \propto r^3$, to answer this question. Two Earth satellites, A and B, orbit at radii of 7.0 × $10^6$ m and 2.8 × $10^7$ m respectively. Which satellite has the longer period of orbit? What is the ratio of orbital radii for the two satellites? What, therefore, is the ratio of the cubes of the orbital radii? What, therefore, is the ratio of the squares of the orbital periods? Finally therefore, what is the ratio of the satellites' orbital periods?

3) What is a geostationary satellite? Describe and explain the orbit of such a satellite. What might such a satellite be used for? With the help of your final expression in question 1, work out the orbital radius of such a satellite. What height is this above the Earth's surface?

4) Suppose we wanted to place a satellite in "jovi-stationary" orbit around Jupiter (the same as geostationary, but around Jupiter, not Earth). What orbital period would it need? What orbital radius would this correspond to?

## Orbital Motion – Teacher's Sheet

## Data required:

$G = 6.67 \times 10^{-11}$ N m$^2$ kg$^{-2}$

mass of the Earth = $6.0 \times 10^{24}$ kg

radius of the Earth = $6.4 \times 10^{6}$ m

mass of Jupiter = $1.9 \times 10^{27}$ kg

radius of Jupiter = $7.2 \times 10^{7}$ m

Jupiter's day length = 10 hours

1) What is the only force acting on a single planet orbiting a star? Write down an expression for this force. If the planet moves in a circular orbit or radius *r*, at constant speed *v*, write down an expression for this speed in terms of the period *T* of the orbit.

Because the orbit is circular, the planet must experience a centripetal force of size $mv^2/r$. Use this fact and the 2 expressions you have written down to prove Kepler's third law, which states that the square of the time period of the planet's orbit is proportional to the cube of the radius of the orbit.

Gravitational attraction $GMm/r^2$.

$v = 2\pi r/T$

| | | |
|---|---|---|
| $GMm/r^2$ | = | $mv^2/r$ |
| $GM/r$ | = | $m\,(2\pi r/T)^2$ |
| $GM/r$ | = | $4\pi^2 r^2 m/T^2$ |
| $T^2$ | = | $(4\pi^2/GM)\, r^3$ |

2) Use Kepler's third law, $T^2 \propto r^3$, to answer this question. Two Earth satellites, A and B, orbit at radii of $7.0 \times 10^6$ m and $2.8 \times 10^7$ m respectively. Which satellite has the longer period of orbit? What is the ratio of orbital radii for the two satellites? What, therefore, is the ratio of the cubes of the orbital radii? What, therefore, is the ratio of the squares of the orbital periods? Finally therefore, what is the ratio of the satellites' orbital periods?

By Kepler's third law, orbital period increases with orbital radius. Thus B has the longer orbital period.

$r_B/r_A = 2.8 \text{ x } 10^7/ 7.0 \text{ x } 10^6 = 4$

$(r_B/r_A)^3 = 4^3 = 64$

By Kepler's third law, $(T_B/T_A)^2 = (r_B/r_A)^3 = 64$

Thus $T_B/T_A$ = square root of 64 = 8

3) What is a geostationary satellite? Describe and explain the orbit of such a satellite. What might such a satellite be used for? With the help of your final expression in question 1, work out the orbital radius of such a satellite. What height is this above the Earth's surface?

A satellite that appears to be stationary over a point on the Earth's equator. The orbit of such a satellite is circular and over the Earth's equator as the satellite's orbital centre is the centre of the Earth, and the only points on Earth that orbit its centre are those on the equator.

Such a satellite might be used for communications, e.g. satellite broadcasting.

$T^2 = (4\pi^2/GM)\ r^3$

$r^3 = (GM/4\pi^2)\ T^2$

$= (6.67 \times 10^{-11} \times 6.0 \times 10^{24} / 4\pi^2) \times (24 \times 60 \times 60)^2$

$= 7.567 \times 10^{22}$ $m^3$ (4sf)

Therefore, $r = 4.23 \times 10^7$ m (3sf)

Height above Earth's surface is $4.23 \times 10^7 - 6.4 \times 10^6 = 3.59 \times 10^7$ m or 36,000 km.

4) Suppose we wanted to place a satellite in "jovi-stationary" orbit around Jupiter (the same as geostationary, but around Jupiter, not Earth). What orbital period would it need? What orbital radius would this correspond to?

10 hours

$r^3 = (GM/4\pi^2)\ T^2$ (see previous question)

$= (6.67 \times 10^{-11} \times 1.9 \times 10^{27} / 4\pi^2) \times (10 \times 60 \times 60)^2$

$= 4.160 \times 10^{24}$ $m^3$ (4sf)

Therefore, $r = 1.61 \times 10^8$ m (3sf)

# TAP 403-2: Using Kepler's third law

Take radius of Earth to be 6 370 km.

1. The radius of a geostationary orbit is 42 200 km. Use this fact together with the constancy of $R^3 / T^2$ to estimate the height above the Earth's surface of a satellite whose circular orbit is completed in 90 minutes. How many times a day would such a satellite orbit the Earth?

2. Low-orbiting Earth satellites usually have orbital periods in the range 90 to 105 minutes. What range of heights does this correspond to?

3. 90 minutes is a typical orbital period for a military reconnaissance satellite, and 100 minutes for a civilian Earth observation satellite. Can you suggest a reason for this difference?

4. Kepler's laws were formulated for *elliptical* orbits (of which the circular orbit is a simple special case). The '*R*' of the third law is the semi-major axis (found as the average of the maximum and minimum distances between a satellite and the body it orbits). You can see how this works by looking at data for Sputnik 1, the first artificial satellite, which was launched on 4 October 1957 and, slowly losing energy due to the effects of atmospheric friction, spiralled back to Earth 3 months later. Complete the following table of data:

| | **4 October 1957** | **25 October 1957** | **25 December 1957** |
|---|---|---|---|
| Orbital period / minutes | 96.2 | 95.4 | 91.0 |
| Minimum height / km | 219 | 216 | 196 |
| Maximum height / km | 941 | 866 | 463 |
| Mean height / km | | | |
| Mean radius / km | | | |
| $R^3 / T^2$<br>three significant figures | | | |

Did the orbit become less elliptical as time passed?

## Practical advice

If Kepler's laws are not in the course specification: you may want to use only a few questions from this set.

## Answers and worked solutions

1.

$$\frac{42\,200\ \text{km}^3}{24\ \text{h}^2} = \frac{R^3}{1.5\ \text{h}^2}$$

so

$$R = \sqrt[3]{\frac{42\,200\ \text{km}^3 \times 1.5\ \text{h}^2}{24\ \text{h}^2}} = 6650\ \text{km.}$$

Therefore

$$h = 6650\ \text{km} - 6370\ \text{km} = 280\ \text{km.}$$

$$\text{number of orbits} = \frac{24\ \text{hours}}{1.5\ \text{hours}} = 16$$

2.

$$\frac{R_{90}^3}{T_{90}^2} = \frac{R_{105}^3}{T_{105}^2}$$

$$\frac{(6650\ \text{km})^3}{(90\ \text{minutes})^2} = \frac{R_{105}^3}{(105\ \text{minutes})^2}$$

$$R_{105}^3 = \frac{(2.9 \times 10^{11}\ \text{km}^3) \times (1.1 \times 10^4\ \text{minutes}^2)}{8.1 \times 10^3\ \text{minutes}^2}$$

$$R = 7330\ \text{km}$$

$$\text{height} = 7330\ \text{km} - 6370\ \text{km} = 960\ \text{km}$$

range is 280 km to 960 km

3. Low orbits give smaller image detail (is it a battlefield tank?); higher orbits give greater coverage and endurance (because there is less atmospheric friction).

4.

| | **4 October 1957** | **25 October 1957** | **25 December 1957** |
|---|---|---|---|
| Orbital period / minutes | 96.2 | 95.4 | 91.0 |
| Minimum height / km | 219 | 216 | 196 |
| Maximum height / km | 941 | 866 | 463 |
| Mean height / km | 580 | 541 | 330 |
| Mean radius / km | 6950 | 6911 | 6700 |
| $R^3 / T^2$ three significant figures | $36 \times 10^6$ | $36 \times 10^6$ | $36 \times 10^6$ |

The Kepler ratio for each case is the same; the deviation from the mean height decreases, so the orbit becomes more like a circle.

Average orbit time was 93.6 minutes. In 3 months (90 days) it made approximately

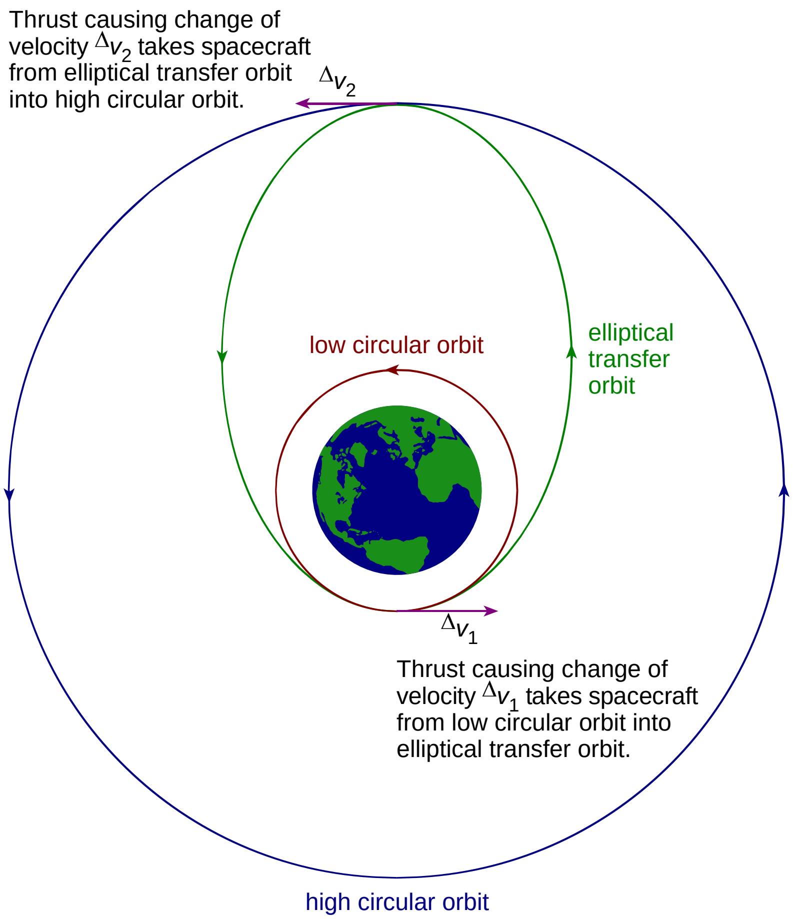
$$\frac{90\ \text{days} \times 24\ \text{hours day}^{-1} \times 60\ \text{min hr}^{-1}}{94\ \text{min}} = 1400\ \text{orbits}$$

## External reference

This activity is taken from Advancing Physics chapter 11, 10D

## TAP 403-3: Changing orbits

Launching a satellite into a circular geostationary orbit is a two-stage operation, involving a lower circular orbit – called a parking orbit – and an elliptical transfer orbit, which is sometimes called a Hohmann orbit.




A satellite is launched into geostationary orbit (radius 42 200 km) via a parking orbit at a height 300 km above Earth's surface. Increases in speed to take the satellite into and out of the elliptical transfer orbit can be regarded as instantaneous.

radius of Earth to be 6370 km.

### A few questions

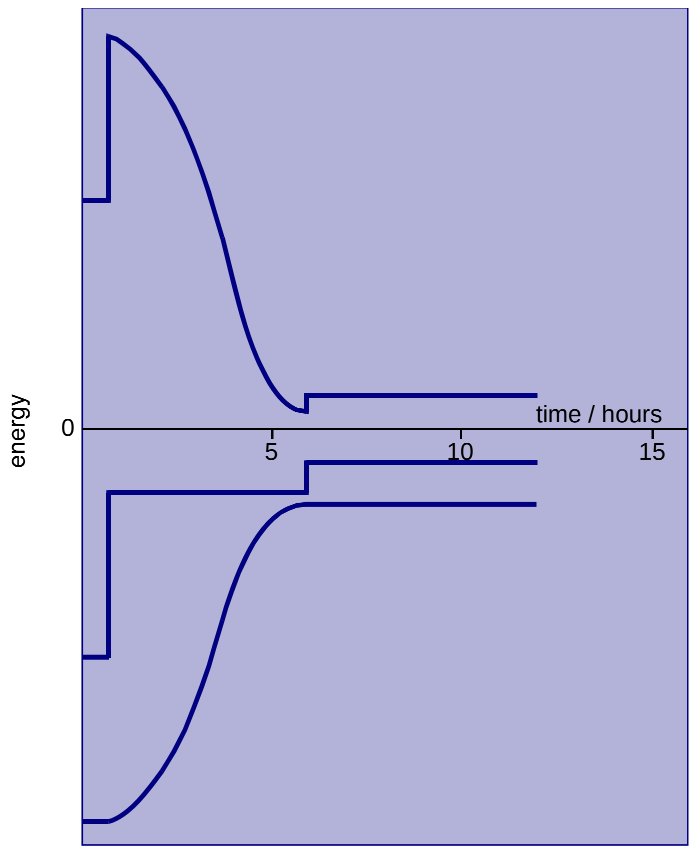
1. Calculate the speed of the satellite in the geostationary orbit.

2. Use Kepler's third law (period$^2 \propto$ mean orbit radius$^3$ ) to calculate the period of the satellite in the parking orbit. Use the period to calculate the speed of the satellite in the parking orbit.

3. Is the kinetic energy of the satellite in the geostationary orbit greater or less than when it is in the parking orbit?

4. Find the average distance for the transfer orbit, by averaging largest and smallest distances. Hence, calculate the time taken for the satellite to travel from parking orbit to geostationary orbit.

5. The graphs show the energy of the satellite before, during and after the transfer between parking and geostationary orbit. Identify which line refers to kinetic energy, which to potential energy and which to total energy.



6. Sketch the path of the satellite around Earth for the 12 h shown.

## Practical advice

This set of questions tries to draw together what students know of elliptical and circular orbits, emphasising the energies needed. Students may need a verbal briefing on what is required for question 6.

## Answers and worked solutions

1.

$$v = \frac{s}{t} = \frac{2\pi \times 42\,200 \text{ km}}{24 \times 60 \times 60 \text{ s}} = 3.07 \text{ km s}^{-1}.$$

2.

$$\frac{T^2}{(1 \text{ day})^2} = \left(\frac{6370 \text{ km} + 300 \text{ km}}{44\,200 \text{ km}}\right)^3$$

$$T = \sqrt{\left(\frac{6670}{42\,200}\right)^3 \times (1 \text{ day})^2} = 0.0628 \text{ days} = 90.5 \text{ minutes}$$

so

$$v = \frac{s}{t} = \frac{2\pi \times 6670 \text{ km}}{90.5 \times 60 \text{ s}} = 7.72 \text{ km s}^{-1}.$$

3. $E_k$ depends on $v^2$ so the kinetic energy is less in the geostationary orbit.

4.

$$R = \frac{6670 \text{ km} + 42\,200 \text{ km}}{2} = 2.44 \times 10^4 \text{ km}$$

and the time for half an orbit will be

$$\frac{T^2}{(1 \text{ day})^2} = \left(\frac{2.44 \times 10^4 \text{ km}}{44200 \text{ km}}\right)^3$$

$$T = \sqrt{\left(\frac{2.44 \times 10^4 \text{ km}}{44200 \text{ km}}\right)^3 \times (1 \text{ day})^2} = 0.44 \text{ day}$$

$$\frac{T}{2} = 0.22 \text{ day} = 5.3 \text{ hr}$$

5. Upper line = kinetic energy; middle line = total energy; lower line = gravitational potential energy.

6. The diagram should reflect the times read from the graph 0.7 h in the parking orbit (half an orbit); 5.3 h in elliptical transfer orbit (half ellipse); 6 h in geostationary orbit (1/4 of orbit). Total 1/2 + 1/2 + 1/4 = 1 1/4 orbits.

## External reference

This activity is taken from Advancing Physics chapter 11, 230D