

TAP 130- 2: Exponential changes

Many naturally occurring changes are exponential. That is, they follow a pattern in which equal steps result in equal fractional changes. The following are just some of many possible examples.

Attenuation of electromagnetic radiation as it travels through a material (e.g. light in a glass fibre). Equal increases in distance, x , result in equal fractional decreases in intensity, I . Mathematically this can be expressed as

$$\frac{\Delta I}{I} = -\mu \Delta x \quad (1)$$

Capacitor discharge through a resistor. The discharge current, I , depends on the pd across the capacitor

$$I = \frac{V}{R}$$

$$I = \frac{\Delta Q}{\Delta t}$$

$$V = \frac{Q}{C}$$

so $\frac{\Delta Q}{\Delta t} = -\frac{Q}{RC}$ (the minus sign indicates that Q is decreasing)

and so

$$\frac{\Delta Q}{Q} = -\frac{\Delta t}{RC} \quad (2)$$

Radioactive decay. The number of nuclei decaying per unit time is proportional to the number of unstable nuclei, N , present in a sample.

$$\frac{\Delta N}{\Delta t} = -\lambda N$$

$$\frac{\Delta N}{N} = -\lambda \Delta t \quad (3)$$

Population growth. The number of new offspring per unit time is proportional to the number of organisms in a population.

$$\frac{\Delta N}{\Delta t} = kN$$

$$\frac{\Delta N}{N} = k \Delta t \quad (4)$$

Equations (1), (2), (3) and (4) all have the same form. They differ only in the symbols used and in the sign of the constant on the right-hand side, which is negative for decay with distance or time, and positive for growth. Each of equations (1), (2), (3) and (4) can be expressed in calculus notation, allowing the intervals to become arbitrarily small. Integration then leads to another form of equation for exponential change. For example, equation (1) becomes

$$\int_{I_0}^{I_x} \frac{dI}{I} = -\mu \int_0^x dx$$

If I_0 is the intensity when $x = 0$, and I_x the intensity after a distance x , integrating between limits gives

$$[\log_e I]_{I_0}^{I_x} = -\mu x$$

$$\log_e \left(\frac{I_x}{I_0} \right) = -\mu x$$

$$\frac{I_x}{I_0} = e^{-\mu x}$$

$$I_x = I_0 e^{-\mu x} \quad (5)$$

Questions

1. Use calculus and equations (2), (3) and (4) to derive equations for the following:
 - (a) charge, Q , on a discharging capacitor as a function of time;
 - (b) number, N , of unstable nuclei as a function of time;
 - (c) number, N , of organisms in a breeding population as a function of time.

In each of equations (1), (2), (3) and (4), the constant on the right-hand side is related to the time or distance over which a quantity changes by a factor of e ($e = 2.718$). For example, from equation (5) we have

$$I_x = \frac{I_0}{e} \quad \text{when} \quad x = \frac{1}{\mu} \quad (6)$$

It is often useful to consider the time or distance over which a quantity halves or doubles. Putting $I_x = I_0/2$ in equation (5) we get

$$\frac{I_0}{2} = I_0 e^{-\mu x}$$

taking reciprocals, cancelling and taking logs to base e gives

$$e^{\mu x} = 2$$

$$\mu x = \log_e (2)$$

so

$$I_x = \frac{I_0}{2} \quad \text{when} \quad x = \frac{\log_e (2)}{\mu} \quad (7)$$

2. Derive expressions for the following:

- (a) The time for the charge on a capacitor (i) to be reduced by a factor of e (ii) to halve.

- (b) The time for the number of unstable nuclei in a sample
 - (i) to be reduced by a factor of e
 - (ii) to halve.

- (c) The time for the number of organisms in a breeding population
 - (i) to rise by a factor e
 - (ii) to double.

Answers and worked solutions

1 (a) $Q = Q_0 e^{-t/RC}$

(b) $N = N_0 e^{-\lambda t}$

(c) $N = N_0 e^{kt}$.

2 (a) (i) $t = RC$

(this is often called the time constant for the discharge and given the symbol τ).

(ii) $t = RC \times \log_e(2)$

(b) (i) $t = \frac{1}{\lambda}$

(ii) $t = \frac{\log_e(2)}{\lambda}$

(this is the half-life of the decay, often given the symbol $t_{1/2}$.)

(c) (i) $t = \frac{1}{k}$

(ii) $t = \frac{\log_e(2)}{k}$

External references

This activity is taken from Salters Horners Advanced Physics, A2, The Medium is the Message, Section MDM, Additional sheet 2